Test 1 CMS/CW 2021-2022 Answer model

1 Make Dirichlet condition homogeneous

Set $u(x) = -3 + \tilde{u}(x)$

Equation for $\tilde{u}$

$$-\frac{d}{dx}\left(\cos x \frac{d\tilde{u}}{dx}\right) + \frac{1}{1+x^2}\tilde{u} = \tan(x) + \frac{3}{1+x^2}$$

$$\tilde{u}(0) = 0, \quad \frac{d\tilde{u}}{dx}(1) + \tilde{u}(1) = 5$$

Forget tildes for a while

$$r_1(u) = -\frac{d}{dx}\left(\cos x \frac{du}{dx}\right) + \frac{1}{1+x^2}u - \tan(x) - \frac{3}{1+x^2}$$

$$r_2(u) = \frac{du}{dx}(1) + u(1) - 5$$

$$\mathcal{V} = \{u \in H^1[0,1] \mid u(0) = 0\}$$

Find $u \in \mathcal{D}$: $(v, r_1(u)) + \alpha\, v_1\, r_2(u) = 0$

for all $v \in \mathcal{D}$

α determined by positivity of bilinear form

$$a(v,u) \equiv \left(v, -\frac{d}{dx}\left(\cos x \frac{du}{dx}\right) + \frac{1}{1+x^2} u\right) + \alpha\, v_1 \left(\frac{du}{dx}(1) + u(1)\right) =$$

$$\overset{p.i}{=} \underbrace{\left(\frac{dv}{dx}, \cos x \frac{du}{dx}\right)} + \underbrace{\left(v, \frac{1}{1+x^2} u\right)} - v \cos(x) \frac{du}{dx}\Big|_0^1 + \alpha\, v_1 \left(\frac{du}{dx}(1) + u(1)\right)$$

$v(0) = 0$

$$= (\quad) + (\quad) + v(1)\left((\alpha - \cos(1)) \frac{du(1)}{dx} + \alpha\, u(1)\right)$$

$$F(v) \equiv \left(v, \tan x + \frac{3}{1+x^2}\right) + \alpha\, 5\, v(1)$$

2 Consider. $a(u,u)$

$$a(u,u) = \underbrace{\Big(\frac{du}{dx}, \underbrace{\cos x}_{>0 \text{ on } [0,1]} \frac{du}{dx}\Big)}_{\geq 0} + \underbrace{\Big(u, \underbrace{\frac{1}{1+x^2}}_{>0} u\Big)}_{\geq 0} + \overbrace{u(1)\underbrace{(\alpha - \cos 1)}_{\text{make zero}} \frac{du}{dx}}^{\text{undeterminate}}$$

$$+ \underbrace{\alpha u^2(1)}_{\text{also } \geq 0}$$

$$\alpha = \cos 1 > 0$$

For this choise $a(u,u) \geq \frac{1}{2}(u,u)$ ~~≥ 0~~ > 0 for $u \neq 0$

~~$a(u,u)$~~

3 $a(u,v) = a(v,u)$

$$a(v,u) = \Big(\frac{dv}{dx}, \cos x \frac{du}{dx}\Big) + \Big(v, \frac{1}{1+x^2} u\Big) + \cos(1)\, v(1)\, u(1)$$

$$= \Big(\frac{du}{dx}, \cos x \frac{dv}{dx}\Big) = \Big(u, \frac{1}{1+x^2} v\Big) + \cos(1)\, u(1)\, v(1)$$

$$= a(u,v)$$

So $A_{ij} = a(\phi_i, \phi_j) = a(\phi_j, \phi_i) = A_{ji}$

matrix symmetric

We know

$a(u,u) > 0 \rightarrow a\left(\sum_{i=1}^{n} c_i \phi_i, \sum_{j=1}^{n} c_j \phi_j\right) > 0$ for $\vec{c} \neq 0$

for $u \neq 0$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \underbrace{a(\phi_i, \phi_j)}_{A_{ij}} = (\vec{c}, A\vec{c})$

> 0 for $\vec{c} \neq 0$

4 Lax - Milgram theorem - see lect. notes

Check conditions:

1 $|a(v,u)| = \left|\left(\frac{dv}{dx} \cos \frac{du}{dx}\right) + \left(v, \frac{1}{(1+x^2)} u\right) + \cos(1)\, u(1)\, v(1)\right|$

$\leq \left\|\frac{dv}{dx}\right\| \left\|\frac{du}{dx}\right\| + \|v\| \|u\| + |u(1)|\,|v(1)|$

$u(1) \leq \left\| \frac{du}{dx} \right\|$

$\leq 2 \left\| \frac{dv}{dx} \right\| \left\| \frac{du}{dx} \right\| + \|v\| \|u\| \leq 2 \|v\|_{H_1} \|u\|_{H_1}$

2 $|F(v)| = \left(v, \tan x + \frac{1}{1+x^2}\right) + 5\cos 1 \; v(1)$

$\leq \|v\| (2 + \tan 1) + 5 |v(1)| \leq$

$\|v\| (2 + \tan 1) + 5 \left\| \frac{dv}{dx} \right\| \leq \max(2 + \tan 1, 5) \|v\|_{H_1}$

3 $a(u,u) \geq \cos 1 \left\| \frac{du}{dx} \right\|^2 + \frac{1}{2} \|u\|^2$

$\geq \min(\cos 1, \frac{1}{2}) \|u\|_{H_1}^2$

5 $A_{ij} = a(\phi_i, \phi_j)$

$\cancel{a(\phi_i, \phi_{i-1}) = \frac{-1}{h^2} \int_{x_{i-1}}^{x_i} \cos x \, dx + \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{x_i - x_{i-1}} \frac{x - x_i}{x_{i-1} - x_i} \frac{1}{1+x^2} dx}$

$\cancel{\text{for } i = 2, \dots, n}$

$$a(\phi_i, \phi_i) = \frac{1}{h^2} \int_{x_{i-1}}^{x_{i+1}} \cos x \, dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 \frac{1}{1+x^2} \, dx$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x - x_{i+1}}{x_i - x_{i+1}} \right)^2 \frac{1}{1+x^2} \, dx$$

for $i = 1, \dots, n-1$

$$a(\phi_n, \phi_n) = \frac{1}{h^2} \int_{x_{n-1}}^{1} \cos x \, dx + \int_{x_{n-1}}^{1} \left(\frac{x - x_{n-1}}{1 - x_n} \right)^2 \frac{1}{1+x^2} \, dx$$

$$+ \cos(1)$$

~~It holds that~~ $\cancel{a(u,v) = a(v,u)}$

~~so~~ $\cancel{A_{ij} = a(\phi_i, \phi_j) = a(\phi_j, \phi_i) = A_{ji}}$

~~so~~ $\cancel{a(\phi_{i-1}, \phi_i) = a(\phi_i, \phi_{i-1})}$

~~And with that the matrix is defined.~~

6. $$\min_{u \in V} \; a(u,u) - 2F(u)$$

where $$a(u,u) = \int_0^1 \cos(x) \frac{du}{dx}^2 + \frac{1}{1+x^2} u^2 \, dx + \cos(1) u^2(1)$$

$$F(u) = \int_0^1 u\left(\tan x + \frac{3}{1+x^2}\right) dx + 5\cos(1) \, u(1)$$

7. According to page 61 of the lecture notes the error is determined by $O(h^{p+1-k})$ where p is degree of the ~~piecewise~~ polynomial used on each element and k the derivative we are looking at

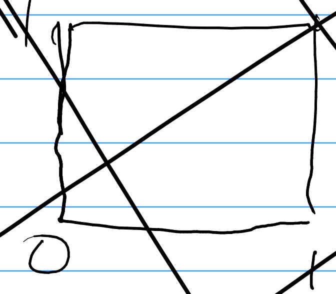
So quadratic elements $\Rightarrow p = 2$

$\|u - u_h\| = O(h^3)$ since there are no derivatives in this norm

$\|u-u_h\|_{H^1} = O(h^2)$ since the first deriv. is in this norm.

8 Lect notes example

9 The linear model holds only for small displacements and stress-strain relation in linear regime.

10 

$u_x = 0 \Rightarrow u = f(y)$

$v_y = 0 \Rightarrow v = g(x)$

$u_y + v_x = 0 \Rightarrow \frac{df}{dy} + \frac{dg}{dx} = 0$

$\Rightarrow \frac{df}{dy} = c \Rightarrow \frac{dg}{dx} = -c$

only constant solution

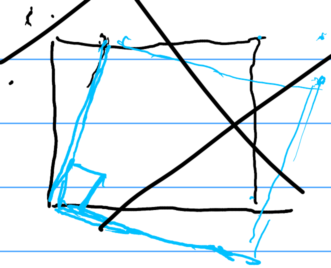
$\rightarrow u = d + cy, \quad v = e - cx$

boundary conditions

$u(x,0) = 0 \rightarrow d = 0$

$v(0,y) = 0 \rightarrow e = 0$

$u(x,y) = cy, \quad v = -cx$



So it is a rotation around 0

$\rightarrow$ for small c